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LINEARIZED THREE COMPONENT MAGNETO-HYDRODYNAMICS

Richard L. Liboff

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Magneto-Fluid Dynamics Division
Courant Institute of Mathematical Sciences
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Errata

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1. p. 6, eq. 1: change $\bar{\rho}_1 Q$ to $\bar{\rho}, Q$.
2. p. 7, eq. 6: change $\bar{U}\bar{Q}$ to $\bar{u}\bar{Q}$.
3. p.10, eq. (a): put brackets about $\frac{\rho}{\rho_o} - \frac{\rho_3}{\rho_3^o}$.

Abstract

A linearized three component MHD theory is given. The three components are two oppositely charged species and a neutral species. Linearized macroscopic equations are presented which in the limit of vanishing magnetic field reduce to fluid dynamical equations, and in the limit of vanishing neutral component give the Lundquist equations. The coupling of the components is frictional. In addition the sound speed that enters is a function of the three equilibrium densities. These equations yield a dispersion relation which is tenth order in the wave speed (ω/k). The decay frequencies (i.e., $-\mathrm{Im}(\omega)$) of all ten modes are calculated to first order terms in the friction, and are found to be all isotropic. In addition all of the modes are stable.

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Linearized Three Component Magneto-Hydrodynamics

I. Introduction and Summary of Results

This analysis is motivated by the fact that most experimental and natural (i.e., astronomical) plasmas are rarely fully ionized. In this paper we will initiate a study of the magneto-hydrodynamic, or low frequency waves which can propagate in a three component plasma in a steady magnetic field. The third component is neutral.

The equations of motion are similar to those stated by Frank-Kamenetskii [1], Watnabe [2] and Schlüter [3]. In the first two works the extension of the "cold", two component plasma investigations [4,5], to the three component case was carried out.

The present work is purely macroscopic. However we are guided by the result of a previous microscopic analysis by the author [6] in which all of the long wavelength phenomena of a two component plasma in a steady magnetic field was uncovered, and formally classified. In this sense, although the included work is strictly adhoc, it is consistent with certain observations partaining to the transition from the microscopic to the macroscopic description in the case of less complicated two component fluid. There it was observed that the six MHD roots [7,8,9] (i.e., normal modes of the linearized Lundquist equations) follow from the more general dispersion relation in the limit $\omega = ak + \ldots$ (a = constant, k = wave number, ω = frequency). Indeed, it is in this formal limiting process that the full equation of motion for the current reduces to the simple $E + u \times B_o = 0$.

In the present analysis the three component macroscopic equations are transformed to equations which describe the two component charged fluid, (in terms of charge and current densities) interacting as one fluid with the neutral component. The set involves equations resembling the Lundquist and neutral fluid [10] equations. The frictional interaction appears as forcing terms in the two momentum equations. A simple thermodynamic treatment yields a form for the three component sound speed. It appears as a function of individual equilibrium mass densities and the equilibrium temperature. The model is such that for vanishing magnetic field, there is only one sound speed, which is believed to be indicative of the true fluid dynamical (i.e. macroscopic) region. The equation of motion for the current is kept in the above infinite conductivity form, which is consistent with the afore-mentioned classification process.

A normal mode analysis indicates that the dispersion relation that this system of equations yields is tenth order in ω . In the limit of vanishing coupling these ten roots are the six MHD roots, the two sound modes, and a degenerate ω^2 factor which is the limiting form that the "particle paths" dispersion relation takes in the present linearization. To first order terms in the friction frequency ν, the decay frequency (i.e., $-\mathrm{Im}(\omega)$) of the three perturbed modes are all: $\nu\rho_3^0/2r_0$ where ρ_3^0 is the equilibrium mass density of the neutral component and r_0 is the total equilibrium mass density. The isotropy of these MHD decay frequencies is a reflexion of the purely frictional coupling of the charged and neutral species; and the null effect of the steady magnetic field in the neutral component.

The decay frequency of the perturbed sound mode is $\nu\rho_0/2r_0$, where ρ_0 is the mass density of the combined charged components. Finally, one of the two "particle path" transverse modes remains as a degenerate ω factor, while the second one decays four times faster than the above sound mode. The degenerate root indicates that to terms of $O(\nu)$ the related perturbation does not decay.

All ten modes are found to be stable.

Acknowledgment

The author is very grateful to Dr. H. Weitzner for many guiding comments pertaining to this work.

II. Analysis.

A. Macroscopic Equations.

The system we will consider is a three component fluid. Two of the components are oppositely charged and the third component is neutral. We wish to examine the small amplitude perturbation of the system about a certain equilibrium state, restricting the investigation to the magneto-hydrodynamical (MHD) region of phenomena, wherein the density of the charged components is large enough so as to insure fluid dynamical reactions. More formally this constraint appears as a limit on the disturbances which will propagate; their frequencies ω being such that $\omega << \omega_o$, where ω_o is the characteristic plasma frequency [11].

The neutral component is similarly constrained in as much as the related phenomena are governed by the fluid dynamical equations, which are a valid description for $\lambda >> \lambda_c$, where λ_c is the mean free path of the neutrals, and λ is the wavelength of the propagating disturbance.

As stated above, the analysis describes the two component charged species interacting as a single fluid with the neutral species. This formulation is brought about through the following transformation of variables. The notation is such that $[\bar{\rho}_1, \bar{\rho}_2, \bar{\rho}_3]$ are the three mass densities where the first two indices relate to the charged components. The bar signifies the "full" state variable, as opposed, say, to the equilibrium value, eg., ρ^o and the perturbation value ρ . The three are related through the equation $\bar{\rho} = \rho_o + \rho$, which in turn defines the perturbation ρ .

In similar manner the macroscopic velocities are $[\bar{u}_1, \bar{u}_2, \bar{u}_3]$. The first two variables of both sets are transformed to the one fluid variable according to:

$$(1) \qquad \bar{u}_1, \bar{u}_2 \longrightarrow \bar{u}, \bar{J}$$

$$\bar{\rho}_1, \bar{\rho}_2 \longrightarrow \bar{\rho}, \bar{Q} \quad ,$$

where

$$\bar{u} = \frac{\bar{\rho}_1 \bar{u}_1 + \bar{\rho}_2 \bar{u}_2}{\bar{\rho}_1 + \bar{\rho}_2} \quad ; \qquad \bar{J} = e \frac{\bar{\rho}_1}{m_1} \bar{u}_1 - e \frac{\bar{\rho}_2 \bar{u}_2}{m_2}$$

$$\bar{\rho} = \bar{\rho}_1 + \bar{\rho}_2 \quad ; \qquad \bar{Q} = e \frac{\bar{\rho}_1}{m_1} - e \frac{\bar{\rho}_2}{m_2} \quad .$$

The neutral variables $(\bar{\rho}_3, \bar{u}_3)$ are kept intact, as are the temperatures $[T_1, T_2, T_3]$.

The equilibrium configuration is given by :

$$J^o = u^o = 0 \quad ; \quad \frac{\rho_1^o}{m_1} = \frac{\rho_2^o}{m_2} \quad ; \quad \rho^o = \rho^o \quad ; \quad \rho_3^o = \rho_3^o$$

$$(3) \qquad r^o = \rho^o + \rho_3^o$$

$$u_3^o = 0 \quad ; \quad T_1^o = T_2^o = T_3^o = T^o \quad ; \quad [E,B]_o = [0,B_o]$$

Note that the new variable r_o is the equilibrium total mass density. As mentioned above, the perturbation is given by,

$$\overline{J} = J^o + J \quad ; \qquad \overline{u} = u^o + u \ , \ \ldots \tag{4}$$

The conservation equations that these (full) variables satisfy are [6] :

The continuity equations

$$\frac{\partial}{\partial t} \overline{\rho} + \nabla \cdot \overline{u} \, \overline{\rho} = 0$$

$$\frac{\partial}{\partial t} \overline{Q} + \nabla \cdot \overline{J} = 0 \tag{5}$$

$$\frac{\partial}{\partial t} \overline{\rho}_3 + \nabla \cdot \overline{\rho}_3 \overline{u}_3 = 0$$

The momentum equations

$$\overline{\rho}\left(\frac{\partial}{\partial t} + \overline{u} \cdot \nabla\right)\overline{u} + \nabla \cdot \overline{p}_{ij} - \overline{Q}(\overline{E} + \overline{u} \times \overline{B}) - \overline{J} \times \overline{B} = \Lambda_1$$

$$\begin{aligned} &\left(\frac{\partial}{\partial t} + \nabla \cdot \overline{u}\right)\overline{J} + \left\{ \overline{Q} \frac{\partial}{\partial t} + (\overline{J} + \overline{UQ}) \cdot \nabla \right\} \overline{u} + \nabla \cdot \overline{R}_{ij} \\ &\qquad - \overline{\rho} \, \frac{e^2}{m_1 m_2} (\overline{E} + \overline{u} \times \overline{B}) - \frac{e(m_1 - m_2)}{m_1 m_2} (\overline{EQ} - \overline{J} \times \overline{B}) = \Lambda_2 \end{aligned} \tag{6}$$

$$\overline{\rho}_3\left(\frac{\partial}{\partial t} + \overline{u}_3 \cdot \nabla\right)\overline{u}_3 + \nabla \cdot \overline{p}_3^{ij} = \Lambda_3 \quad .$$

The tensor $\overline{p}_{ij}$ is the pressure term, and the variable $\overline{R}_{ij}$ represents the "J-Flux" tensor [6] . The terms Λ_i are functionals of the collision integrals. By conservation of momentum in a collision $\Lambda_1 + \Lambda_3 = 0$.

The above equations are standard moment equations. They are written down here to indicate the effect that the formal change of variables has on the equations involved. The succesive higher order moment equations (e.g., pressure, heat flow, ...) would be of the same form, the first two in each case relating to the charged components and the third to the neutral component. It is clear that the formal change of variables yields equations which describe the charge components as a coupled fluid and the neutral component as another interacting fluid. The suggestion is something less than fortuitous, as the terms which enter into the charge component equations are physically relevant (e.g., J, Q). These are the variables which interact with the fields in recognizable manner, which in turn is the major requirement that a variable must satisfy in an ad-hoc theory.

If the above equations are linearized about the equilibrium state as given by equation (3), and the fluid dynamical constraint is imposed [6], then the lowest order terms (in the wave number k) appear as (including Maxwell's equations) :

$$
\begin{aligned}
&\dot{\rho} + \rho_0 \nabla \cdot u = 0 \\
&\dot{\rho}_3 + \rho_3^0 \nabla \cdot u_3 = 0 \\
(7)\qquad &\rho_0 \dot{u} + a^2 \nabla \rho = J \times B_0 - \frac{\rho_0 \rho_3^0 \nu}{r_0} (u - u_3) \\
&\rho_3^0 \dot{u}_3 + a^2 \nabla \rho_3 = \frac{\rho_0 \rho_3^0\ \nu}{r_0} (u - u_3)
\end{aligned}
$$

$$
E + u \times B_0 = 0 \quad ; \qquad \nabla \cdot B = 0
$$

$$
\nabla \times B = \mu_0 J \quad ; \qquad \nabla \times E = - \dot{B}
$$

The collision forms Λ_1 and Λ_3 are replaced by the momentum exchange terms : $-$ and $+$ $(\rho_0 \rho_3^0 / r_0)\nu(u-u_3)$, respectively. The single sound speed a , that enters, is consistent with simple thermodynamic considerations. First we recall the adiabatic formula $P = Ar^\gamma$, where P is the total pressure, Σp_i , r is the total mass density $\Sigma \rho_i$, A is a constant, and $\gamma = (d+2)/d = 5/3$ for $d = 3$, i.e., three degrees of freedom. Each component obeys an equation of state $p_i = \rho_i R_i T_0$ (the thermodynamic equilibrium is such that $T_i = T_0$) . It follows that $P = r\, \overline{R}\, T_0$ where,

$$
(8) \qquad \overline{R} = \frac{\rho_1 R_1 + \rho_2 R_2 + \rho_3 R_3}{r}
$$

For the sound speed a , we have :

$$
(9) \qquad a^2 = \frac{\partial p}{\partial r} = \gamma \frac{p}{r} = \gamma\, \overline{R}\, T_0 = \frac{5}{3} \overline{R}\, T_0 \, .
$$

In the specific configuration under considerations:

$$(10) \qquad \bar{R} = \kappa\, [2n_1^o + n_3^o] \Big/ [Mn_1^o + m_3 n_3^o] \ ,$$

where we have written M for $m_1 + m_2$, and have recalled that the equilibrium configuration is charge neutral, so that the equilibrium number densities n_1^o , satisfy $n_1^o = n_2^o$. κ is Boltzmann's constant.

Before examining the linearized system (7) for plane wave solutions let us first consider two limiting cases.

(a) $\underline{B_o \longrightarrow 0}$

In this instance the system (7) reduces to

$$r_o \dot{U} + a^2 \nabla r = 0$$

$$\dot{u} - \dot{u}_3 + a^2 \nabla \left\{ \frac{\rho}{\rho_o} - \frac{\rho_3}{\rho_3^o} \right\} = - \nu (u - u_3)$$

$$\dot{r} + r_o \nabla \cdot U = 0$$

$$\frac{\partial}{\partial t} \ \frac{\rho}{\rho_o} - \frac{\rho_3}{\rho_3^o} \ + \nabla \cdot \left\{ u - u_3 \right\} = 0 \ .$$

Capital U represents the mass averaged velocity

$\dfrac{\rho_1^o u_1 + \rho_2^o u_2 + \rho_3^o u_3}{r_o}$, while r represents the total mass density $\rho_1 + \rho_2 + \rho_3$.

In the limit $B_o \longrightarrow 0$, the coupling to Maxwell's equations, $J \times B_o$, vanishes. Furthermore since $B_o = 0$, Ohm law gives $E = 0$, which in turn implies that B is constant in time. The only remaining relevant field equation is $\nabla \times B = J$, which is decoupled from the set (11).

The set (11) yields the dispersion relation:

$$\omega^2(\omega^2-a^2k^2)(\omega+i\nu)^2[\omega(\omega+i\nu) - a^2k^2] = 0 \quad . \tag{12}$$

The first and third factors give the eigenvalues of the transverse components of U , and $u-u_3$, respectively. For U say, these components may be written $k \times U$, and $k \times (U \times k)$. The second factor gives eigenvalues for the longitudinal component $k \cdot U$, of U ; and for the total mass density r . The last factor relates to the longitudinal component of the difference flow $(u-u_3)$, and the difference mass density $[(\rho/\rho_o) - (\rho_3/\rho_3^o)]$. It should be noted that in the space homogeneous limit $k \longrightarrow 0$, this last factor yields the two roots $\omega = 0$ and $i\omega = \nu$;which relate respectively first to a state wherein the perturbation difference in mass densities remains constant, and second to state wherein the two flow velocities relax to each other. The decay time for this latter phenomena is ν^{-1} . More generally the third factor relates to similar relation of the transverse vibration.

(b) $\rho_3^0 \longrightarrow 0$

In this second limit the system (7) reduces to :

$$\dot{\rho} + \rho_0 \nabla \cdot u = 0$$

$$\rho_0 \dot{u} + a^2 \nabla \rho = J \times B_0 \tag{13}$$

$$E + u \times B_0 = 0 \ ,$$

together with Maxwell's equations. This aggregate of equations is recognized to be the standard linearized Lundquist set, which yield the so called Fast, Slow and Alfven modes [7-9].

B. Dispersion Relations

We will now examine the full system (7) for normal mode solutions. First we note that the level of description of the included analysis is given by the dimensionality of the state vector Γ,

$$\Gamma = \left\{ E, B, u, J, \rho, u_3, \rho_3 \right\} \tag{14}$$

The independent components of Γ are as follows: three components of u , two components of B , three components of u_3 , together with ρ and ρ_3 . These components individually satisfy first order time differential equations. The time dependance of the remaining variables are determined in terms of these independent variables.

The plane wave transform Γ^* of the state vector Γ is defined as:

$$\Gamma(x,t) = \Gamma^*(\omega,k)e^{i(k\cdot x-\omega t)} \tag{15}$$

In terms of this state vector Γ, the system appears as:

$$A_{\ell j}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \nu ;\ \rho_o, \rho_3^o, a^2, B_o, c, \varepsilon_o\right)\Gamma_j = 0 \tag{16}$$

The transform of the above equation appears as (writing only the dependence of the first three parameters):

$$A_{\ell j}(\omega, k, \nu)\Gamma_j^* = 0 , \tag{17}$$

which yields the dispersion relation

$$D = |A_{\ell j}(\omega,k, \nu)| = 0 . \tag{18}$$

The function D is a tenth order polynomial in ω. If we choose $B_o = (0,0,B_o)$ and $k = (k_x,0,k_z)$, then D appears as (after some elimination):

$$D = |A| = \omega^{-2} \left| \begin{array}{ccc|ccc} -\omega^2+a^2k_z^2+A^2k^2 - i\bar{\nu}\omega & 0 & a^2k_xk_z & i\bar{\nu}\omega & 0 & 0 \\ 0 & k_z^2A^2-\omega^2 - i\bar{\nu}\omega & 0 & 0 & i\bar{\nu}\omega & 0 \\ k_xk_za^2 & 0 & -\omega^2+k_z^2a^2 - i\bar{\nu}\omega & -i\bar{\nu}\omega \dfrac{k_x}{k_z} & 0 & i\bar{\nu}\omega \\ \hline i\hat{\nu}\omega & 0 & 0 & -\omega(\omega+i\hat{\nu}) & 0 & a^2k_xk_z \\ 0 & i\hat{\nu}\omega & 0 & 0 & -\omega(\omega+i\hat{\nu}) & 0 \\ 0 & 0 & -i\hat{\nu}\omega & \omega(\omega+i\hat{\nu}) \dfrac{k_x}{k_z} & 0 & a^2k_z^2 - \omega(\omega+i\hat{\nu}) \end{array} \right| \tag{19}$$

where,

$$(20) \qquad \bar{\nu} = (\rho_3^o/r_o)\nu \quad ; \quad \hat{\nu} = (\rho_o/r_o)\nu$$

We will now proceed to find the roots of the dispersion relation (18) to terms of $O(\nu)$. To these ends D is expanded in a Taylor series about $\nu = 0$:

$$(21) \qquad D(\nu) = D(0) + \nu D'(0) + \frac{\nu^2}{2} D''(0) + \ldots ,$$

where D appears formally as :

$$(22) \qquad D = \sum_{(e_1,e_2,e_3,\ \ldots\ e_6)} (-)^p d_{1e_1} d_{2e_2} d_{3e_3} \ \ldots \ d_{6e_6}$$

where p is the order of the permutation $P(e_1,e_2, \ldots)$. It follows that the derivative of D is given by :

$$(23) \qquad D' = \sum (-)^p d'_{1e_1} d_{2e_2} d_{3e_3} \ldots + \sum (-)^p d_{1e_1} d'_{2e_2} d_{3e_3} \ldots + \sum (-)^p d_{1e_1} d_{2e_2} d'_{3e_3} \ \ldots + \ldots$$

The n^{th} sum is the original determinant with the n^{th} row replaced by the derivative of that row. If these determinants are evaluated and then ν is set equal to zero it is easily verified that the only terms that contribute to $D'(0)$ are the diagonal 3 \ 3 subdeterminants. It follows that for purposes of evaluating $D(\nu)$ to terms of $O(\nu)$, we may set:

$$D = k^{10} D_o D_1 \qquad , \tag{24}$$

where

$$D_o = [v^4 - v^2(a^2+A^2) + a^2A_z^2 - \bar{v}_\nu v(a^2+A^2-2v^2)](A_z^2 - v^2 - \bar{v}_\nu v)$$

(25)

$$D_1 = (v + \hat{v}_\nu)^2[a^2 - v(v + \hat{v}_\nu)] \; .$$

The new variables are defined as:

$$v = \omega/k \quad ; \quad A^2 = B_o^2/\mu_o\rho_o \quad ; \quad A_z^2 = A^2\cos^2\theta$$

(26)

$$\hat{v}_\nu = \frac{i\hat{\nu}}{k} \; ; \quad \bar{v} = i\,\frac{\bar{\nu}}{k} \quad ; \quad v_\nu = \frac{i\nu}{k} \quad ; \quad k_z = k\cos\theta$$

$$\frac{\partial\hat{v}_\nu}{\partial v_\nu} = \frac{\rho_o}{r_o} \quad ; \quad \frac{\partial\bar{v}_\nu}{\partial v_\nu} = \frac{\rho_3^o}{r_o}$$

The Taylor series of $D(v_\nu)$ about $v_\nu = 0$ now appear as:

$$k^{-10} D(v_\nu) = D(0) + \left(D_1 \frac{\partial D_o}{\partial \bar{v}_\nu} \frac{\partial \bar{v}_\nu}{\partial v_\nu} + D_o \frac{\partial D_1}{\partial \bar{v}_\nu} \frac{\partial \hat{v}_\nu}{\partial v_\nu}\right)_{\nu_2 = 0} v_\nu + \dots ,$$

$$\left. \frac{\partial D_o}{\partial \bar{v}_\nu} \frac{\partial \bar{v}_\nu}{\partial v_\nu} \right|_{v_\nu = 0} = - \frac{\rho_3^o}{r_o} v[2a^2A_z^2 + A_z^2A^2 - 2v^2A^2 - 2v^2A_z^2 - 2v^2a^2 + 3v^4]$$

$$\left. \frac{\partial D_1}{\partial \hat{v}_\nu} \frac{\partial \hat{v}_\nu}{\partial v_\nu} \right|_{v_\nu = 0} = \frac{\rho_o}{r_o} v \; [2a^2 - 3v^2] \qquad .$$

The unperturbed sub-determinants $D_1(0)$ and $D_o(0)$ are:

$$D_o(0) = [v^4 - v^2(a^2 + A^2) + a^2A_z^2] \; (A_z^2 - v^2)$$

(28)

$$D_1(0) = (a^2 - v^2)v^2 \qquad ,$$

where we recall

(29) $$D(0) = D_o(0) \; D_1(0)$$

The roots of $D_o(0)$ are the six modes of the Lundquist [8] relation, viz :

(30) $$D_o(0) = (A_z^2 - v^2) \; (v^2 - A_+^2) \; (v^2 - A_-^2) \qquad .$$

These three sets of roots are, respectively:

(1) The Alfvén wave :

$$v^2 = A_z^2 = A^2 \cos^2 \theta \quad , \tag{31}$$

(2) The "Fast" wave :

$$v^2 = A_+^2 = \left(\frac{a^2+A^2}{2}\right) + \sqrt{\frac{a^2+A^2}{2}^{\,2} - a^2A_z^2} \tag{32}$$

(3) The "Slow" wave :

$$v^2 = A_-^2 = \left(\frac{a^2+A^2}{2}\right) - \sqrt{\frac{a^2+A^2}{2}^{\,2} - a^2A_z^2} \tag{33}$$

The roots of $D_1(0)$ are the sound mode and the degenerate transverse mode.

It remains to uncover the extension of these roots to terms of $O(v_\nu)$. All of these are of the form :

$$v_i = V_i + \sigma_i v_\nu \quad , \tag{34}$$

where V is the ten component vector :

$$V = (\pm A_z, \pm A_+, \pm A_-, \pm a, \pm 0) \quad . \tag{35}$$

Substitution of (34) into the dispersion relation (24) and retaining terms to $O(v_\nu)$ give the desired results.

First we note that the $D_o(0)\,D_1(0)$ factor involves terms of the form $v_i^2 - V_i^2$, so that the difference between the two substitutions $v_i^{(\pm)} = \pm V_i + \sigma_i^{(\pm)} v_\nu$ into this factor is a change in sign. The remaining first derivatives terms accept only the lower $\pm V_i$, part of $v_i^{(\pm)}$ since σ part would give an $O(v_\nu^2)$ term. In addition both of the higher order derivatives terms are of the form $vf(v^2)$, so that the difference between the related two mirror substitutions $\pm V_i$, also involves only a change in sign. It follows that the same equation results for σ_i^+ and σ_i^-, so that it is consistent to write :

$$(36) \qquad v^{\pm} = \pm V_i + \sigma_i v_\nu$$

This observation together with the observation that $\sigma_i < 0$, is merely a consequence of the stability of the equilibrium configuration.

The said substitutions yield the following results:

I : The MHD Roots

(a) Alfvén Modes

$$(37) \qquad v_1 = \pm A_z + \sigma_1 v_\nu , \qquad \sigma_1 = - \rho_3^o / 2r_o$$

(b) Fast and Slow Modes

$$v^{+}_{2,\pm} = \pm A_{+} + \sigma^{+}_{2} v_{\nu}$$

(38)

$$v^{-}_{2,\pm} = \pm A_{-} + \sigma^{-}_{2} v_{\nu}$$

where :

$$(39) \quad 2\sigma^{(\pm)} = \frac{\rho^{o}_{3}}{r_{o}} \left[\frac{A^{2}_{z}(2a^{2}+A^{2}) - 2A^{2}_{\pm}(A^{2}_{z}+A^{2}+a^{2}) + 3A^{4}_{\pm}}{(A^{2}_{z} - A^{2}_{\pm})\,(A^{2}_{\pm} - A^{2}_{\mp})} \right] \quad .$$

If the values for A_z , $A_\pm$ as given by (31-33) are substituted into the above, one finds that*

$$(40) \qquad \sigma^{\pm}_{2} = -\rho^{o}_{3} \big/ 2r_{o} \quad .$$

The decay times of the three MHD waves are all equal and isotropic.

II The Neutral Gas Roots

The extended sound modes are written

$$(41) \qquad v^{\pm}_{3} = \pm a + \sigma_{3} v_{\nu} \ ,$$

which when substituted into equation (24) yields :

---

* This observation is due to N. Ainbender, of the CIMS computational group.

$$\sigma_3 = -\rho_o/2r_o \tag{42}$$

One of the remaining transverse modes remains zero to $O(v_\nu)$ (i.e. the perturbation remains in a steady state) while the remaining mode gives:

$$v_4 = -2v_\nu \rho_o/r_o \quad , \tag{43}$$

a very fast decaying root.

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